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CHAPTER 9
ROLL COUPLING

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9.1 INTRODUCTION

Divergence experienced during rolling maneuvers has frequently been referred to as "inertial coupling." This leads to a misconception of the problems involved. The divergence experienced during rolling maneuvers is complex because it involves not only inertial properties, but aerodynamic ones as well. The material in this chapter is intended to offer a physical explanation of the more important causes of roll coupling.

Coupling results when a disturbance about one aircraft axis causes a disturbance about another axis. An example of uncoupled motion is the disturbance created by an elevator deflection. The resulting motion is restricted to pitching motion, and no disturbance occurs in yaw or roll. An example of coupled motion is the disturbance created by a rudder deflection. The ensuing motion will be some combination of both yawing and rolling that results in coupling problems large enough to threaten the structural integrity of the aircraft.

There are numerous contributions to the roll coupling characteristics of an aircraft. Only three will be considered here:

Inertial Coupling

The I_{xz} Effect

Aerodynamic Coupling

These effects occur simultaneously in a complex fashion. Therefore, divergence cannot be predicted by analyzing these effects separately. The complicated interrelationship of these parameters can best be seen by analyzing the aircraft equations of motion.

$$\text{Roll } \dot{p} = \frac{\Sigma L}{I_x} - qr\frac{(I_z - I_y)}{I_x} + (\dot{r} + qp)\frac{I_{xz}}{I_x} \tag{9.1}$$

$$\text{Pitch } \dot{q} = \frac{\Sigma M}{I_y} - pr\frac{(I_x - I_z)}{I_y} - (p^2 - r^2)\frac{I_{xz}}{I_y} \tag{9.2}$$

$$\text{Yaw } \dot{r} = \frac{\Sigma N}{I_z} - pq\,\frac{(I_y - I_x)}{I_z} - (qr - \dot{p})\,\frac{I_{xz}}{I_z} \quad (9.3)$$

$$\text{Drag } \dot{u} = \frac{\Sigma F_x}{m} - qw + rv \quad (9.4)$$

$$\text{Lift } \dot{w} = \frac{\Sigma F_z}{m} - pv + qu \quad (9.5)$$

$$\text{Side } \dot{v} = \frac{\Sigma F_y}{m} - ru + pw \quad (9.6)$$

Consider Equations 9.1 - 9.3, derived from moment equations. In each case the first term on the right-hand side of the equations represents the aerodynamic contribution, the second term the inertial contribution, and the third term the I_{xz} effects. It can be seen that these three contributions to roll coupling can combine to adversely or proversely affect pitch, roll, and yaw acceleration, and thus the tendency for the aircraft to diverge.

"Divergence" in roll coupling is characterized by a departure from the intended flight path that will result in either loss of control or structural failure. As defined, this " divergence" is what we are concerned with in roll coupling. Smaller roll coupling effects that do not result in divergence will not be considered. It should be noted that divergence about any one axis will be closely followed by divergence about the others.

9.2 INERTIAL COUPLING

Inertial coupling did not become a problem until the introduction of the century series aircraft. As the fighter plane evolved from the conventional design of the P-47 and P-51, through the first jet fighter, the F-80, and then to the F-100 and other century series aircraft, there was a steady change in the weight distribution. During this evolution, more and more weight was concentrated in the fuselage as the aircraft's wings grew thinner and shorter. This shift of weight caused relationships between the moments of inertia t

change. As more weight was concentrated along the longitudinal axis, the moment of inertia about the x axis decreased relative to the moments of inertia about the y and z axis. This phenomena increases the coupling between the lateral and longitudinal equations. This can be seen by examining Equation 9.2

$$\dot{q} = \frac{\Sigma M}{I_y} - pr\,\frac{I_x - I_z}{I_y} - (p^2 - r^2)\,\frac{I_{xz}}{I_y} \tag{9.2}$$

As I_x becomes much smaller than I_z, the moment of inertia difference term $(I_x - I_z)/I_y$ can become large. If a roll rate is introduced, the term $pr\,(I_x - I_z)/I_y$ may become large enough to cause an uncontrollable pitch acceleration.

Modern fighter design is characterized by a long, slender, high density fuselage with short, thin wings. This results in a roll inertia which is quite small in comparison to the pitch and yaw inertia. The more conventional low speed aircraft may have a wingspan greater than the fuselage length and a great deal of weight concentrated in the wings. A comparison of these configurations is presented in Figure 9.1.

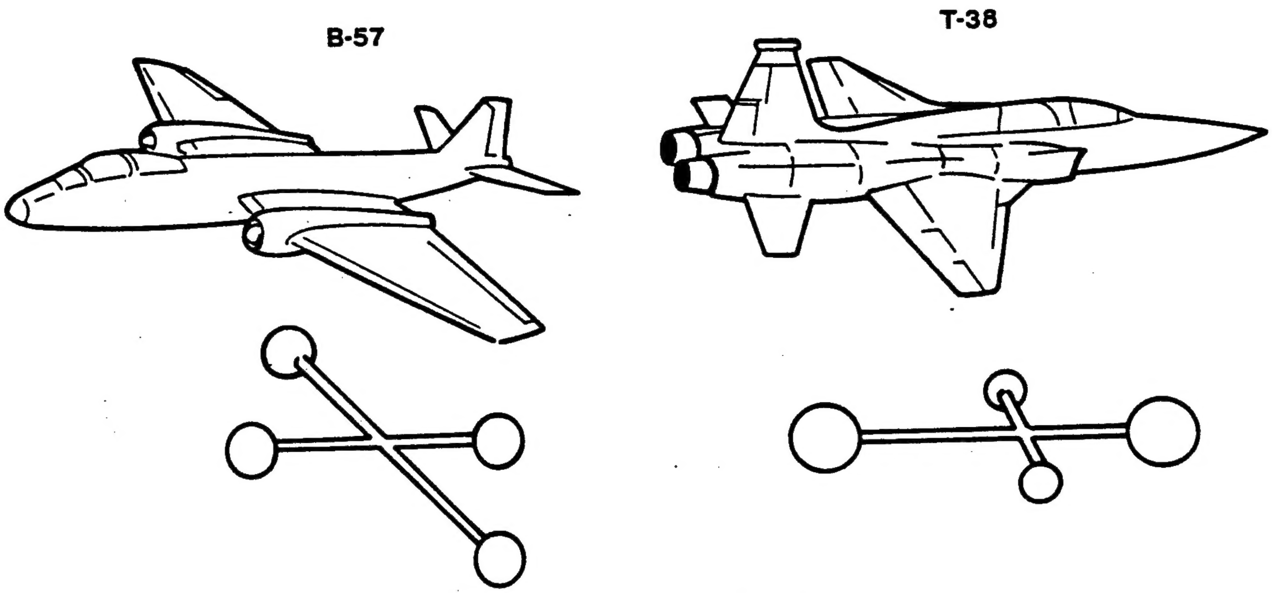


FIGURE 9.1. CONVENTIONAL AND MODERN AIRCRAFT DESIGN

The mass distribution of these aircraft can be represented by a pair c dumbbells. The axis with the larger dumbbell will tend to align itself wit the plane that is perpendicular to the roll axis. Therefore, inertia coupling for the B-57 is different than that of the T-38. A roll will hav little effect on angle of attack for the B-57 and increase it for the T-38.

The conventional B-57 design presents considerable resistance to rotatic about the x axis and does not generate high roll rates. On the other hanc the T-38 design presents a relatively small resistance to rotation about the axis and attains high rates of roll. High roll rates enhance the tendenc toward inertial coupling.

This analysis of inertial coupling will consider rolls about tw different axes, the inertial axis and the aerodynamic axis. The inertial axi is formed by a line connecting the aircraft's two "centers of inertia" a shown in Figure 9.2. The aerodynamic axis is the stability x axis firs introduced in the investigation of the left-hand side of the equations c motion. It is merely the line of the relative wind. Aircraft rotation in roll is generally assumed to be about this axis. To visualize this, recal that to produce a rolling moment a differential in lift must be created on th wings. For the time being, let us assume that the aircraft will roll abou the relative wind, or aerodynamic axis.

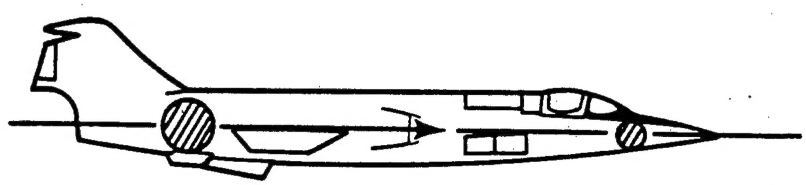


FIGURE 9.2. AIRCRAFT INERTIAL AXIS

First, consider a roll when the aerodynamic and inertial axes ar coincident as illustrated in Figure 9.3. In this case, there is no forc created by the centers of inertia that will cause the aircraft to be diverte

from its intended flight path, and no inertial coupling results. Now, observe what happens when the inertial axis is displaced from the aerodynamic axis.

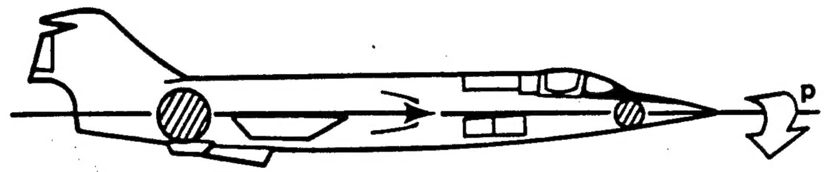


FIGURE 9.3. AERODYNAMIC AND INERTIAL AXES COINCIDENT

As the aircraft is rotating about the aerodynamic axis, centrifugal force will act on the centers of inertia. Remembering that centrifugal force acts perpendicular to the axis of rotation, it can be seen that a moment will be created by this centrifugal force. For the case depicted in Figure 9.4, where the aerodynamic axis is depressed below the inertial axis, a pitch up will result. Conversely, if the aerodynamic axis is above the inertial axis, a pitch down will result.

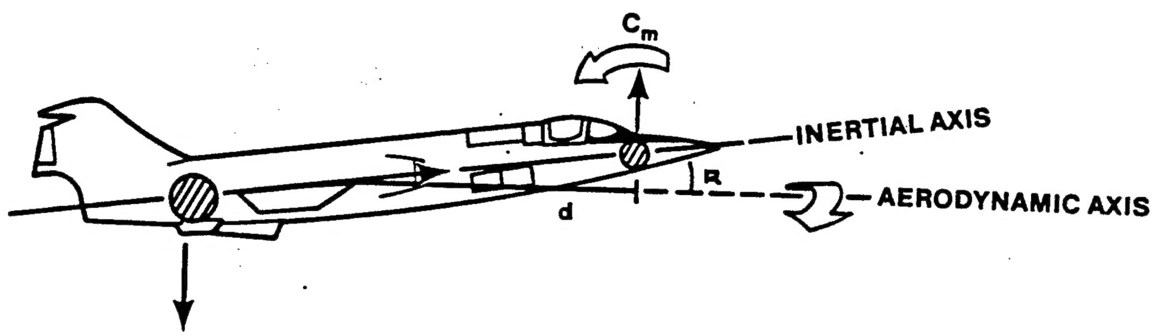


FIGURE 9.4. AERODYNAMIC AND INERTIAL AXES NONCOINCIDENT

To appreciate the magnitude of the moment thus developed, refer to Figu
9.4 and consider the following:

$$\text{Centrifugal Force (CF)} = \frac{mV^2_{Tangential}}{R} \qquad (9.$$

$$V_{Tangential} = R\omega = Rp \qquad (9.$$

Therefore,

$$CF = mRp^2$$

The moment created by this centrifugal force is

$$M = (CF)(d) = mRp^2d \qquad (9.$$

For modern designs, m is large. Also, R will be larger for a low aspect rat wing. (The aircraft will operate at a higher angle of attack.) As previous discussed, p will be large. For long, dense fuselages, d will be larg Thus, the moment created by inertial coupling will be large.

9.3 THE I_{xz} EFFECT

Three products of inertia (I_{xy}, I_{yz}, and I_{xz}) appear in the equations motion for a rigid aircraft. By virtue of symmetry, I_{xy} and I_{yz} are bo equal to zero. However, the product of inertia I_{xz} can be of an appreciab magnitude and can have a significant effect on the roll characteristics of aircraft.

The parameter, I_{xz}, can be thought of as a measure of the nonuniformi of a mass distribution along the x axis, and the mass of the aircraft can considered to be concentrated on this axis. The axis about which $I_{xz} = 0$ defined as the inertial axis.

The I_{xz} parameter is a measure of how the inertial axis is displaced fr the aircraft x axis. A typical aircraft design can be represented by t centers of mass in the xz plane designated m_1 and m_2 in Figure 9.5. It can seen that if the aircraft is rolled about the x axis, a pitch down wi result. The inertial pitching moment (up or down) generated by a roll is

function of (1) the axis about which the roll is performed and (2) the inclination of the inertial axis with respect to the roll axis. A roll about the aerodynamic axis in Figure 9.6a will produce a pitch down while the same roll in Figure 9.6b will produce a pitch up. Thus, when an aircraft is rolled about an axis which differs from its inertial axis, pitching moments develop which tend to cause the aircraft to depart from its intended flight path. Depending on its orientation, the I_{xz} parameter modifies the effect of inertial coupling.

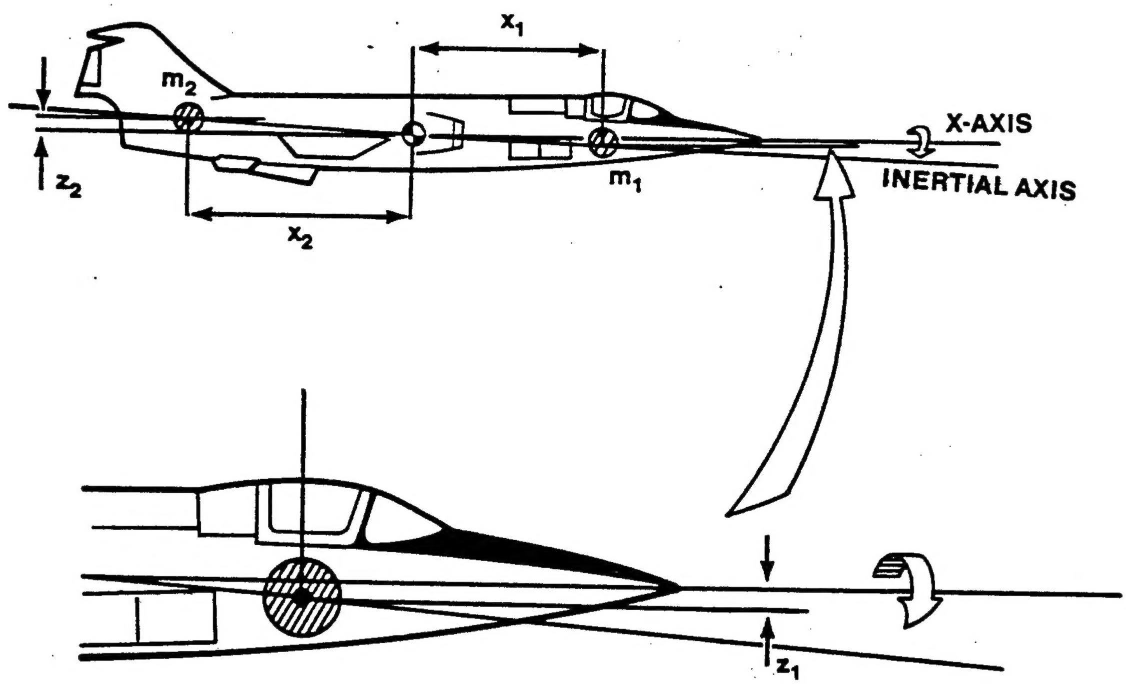


FIGURE 9.5. INERTIAL AXIS BELOW AERODYNAMIC AXIS

To appreciate the magnitude of this parameter, consider Figure 9.5. From Equation 9.9, the moment produced by the forward center of mass is,

$$M_1 = (C.F.)(x_1) = m_1 x_1 p^2 z_1 \tag{9.10}$$

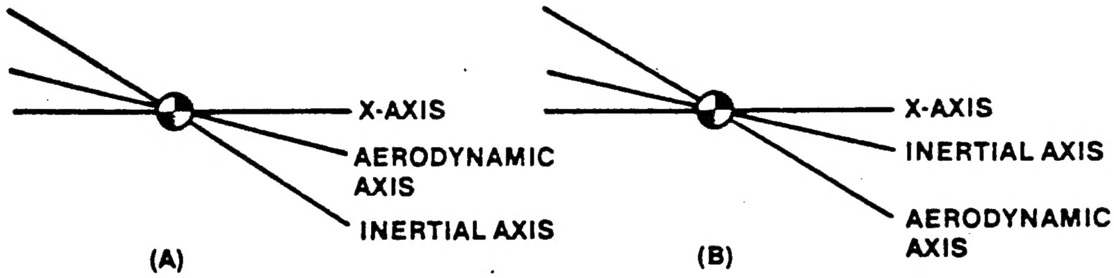


FIGURE 9.6. THE I_{xz} EFFECT

Similarly, the moment produced by the aft center of mass is

$$M_2 = m_2 x_2 p^2 z_2 \quad (9.11)$$

The total pitch moment is therefore

$$M_T = M_1 + M_2 = m_1 x_1 p^2 z_1 + m_2 x_2 p^2 z_2 \quad (9.12)$$

$$M_T = p^2 (m_1 x_1 z_1 + m_2 x_2 z_2) \quad (9.13)$$

But for a simplified system

$$I_{xz} = m_1 x_1 z_1 + m_2 x_2 z_2 \quad (9.14)$$

Therefore,

$$M_T = p^2 I_{xz} \quad (9.15)$$

Thus, it can be seen that the magnitude of the pitching moment depends on the roll rate and the magnitude of the I_{xz} parameter relative to the roll axis. To differentiate between roll coupling and the I_{xz} effect, realize that inertial coupling occurs when the aircraft is not rolled about its inertial axis and the I_{xz} parameter, depending on its orientation, either amplified or reduces the magnitude of inertial coupling.

9.4 AERODYNAMIC COUPLING

This analysis of roll coupling is not concerned with all aerodynamic coupling terms C_{n_p}, $C_{n_{\delta_a}}$, C_{l_r}, $C_{l_{\delta_r}}$, etc. . Only the "kinematic coupling" aspects of aerodynamic coupling will be considered.

Kinematic coupling is the actual interchange of α and β during a rolling maneuver. This interchange is an important means by which the longitudinal and lateral motions are capable of influencing each other during a rapid roll.

To understand how this interchange of α and β occurs, consider Figure 9.7. In this figure the aircraft is assumed to have either infinitely large inertia or negligible stability. Thus it will roll about its inertial axis. In (I) the aircraft initiates a roll from a positive angle of attack. In (II) the initial angle of attack is converted to a positive sideslip angle of equal magnitude after 90° of roll. In (III) the aircraft has again exchanged β and α and after 180° of roll has an angle of attack equal in magnitude but opposite in sign to the original α. The interchange continues and in (IV) this $-\alpha$ is converted to $-\beta$.

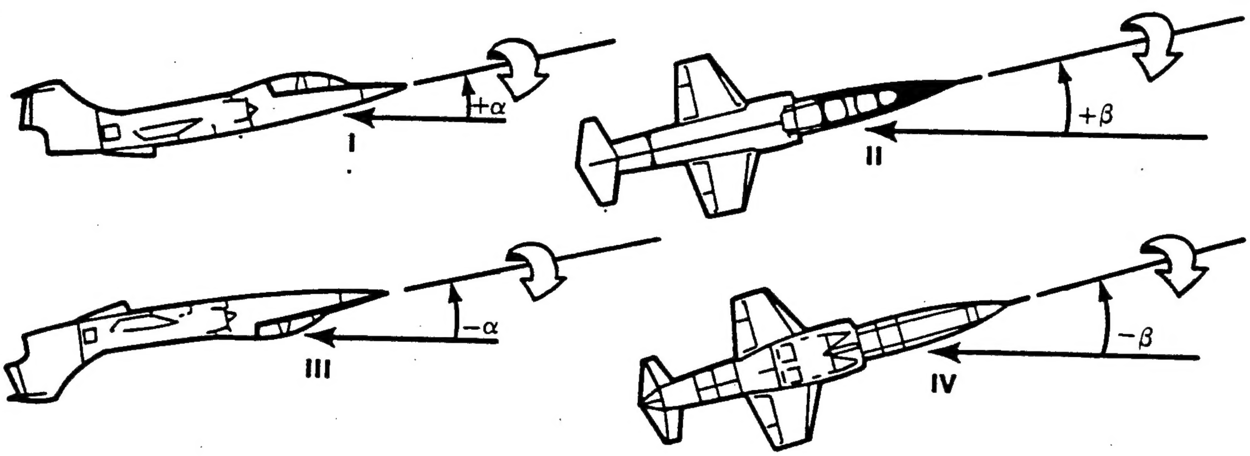


FIGURE 9.7. KINEMATIC COUPLING. ROLLING OF AN AIRCRAFT WITH INFINITELY LARGE INERTIA OR NEGLIGIBLE STABILITY IN PITCH AND YAW

Next, consider an aircraft with infinitely large stability in pitch and yaw or negligible inertia. Refer to Figure 9.8.

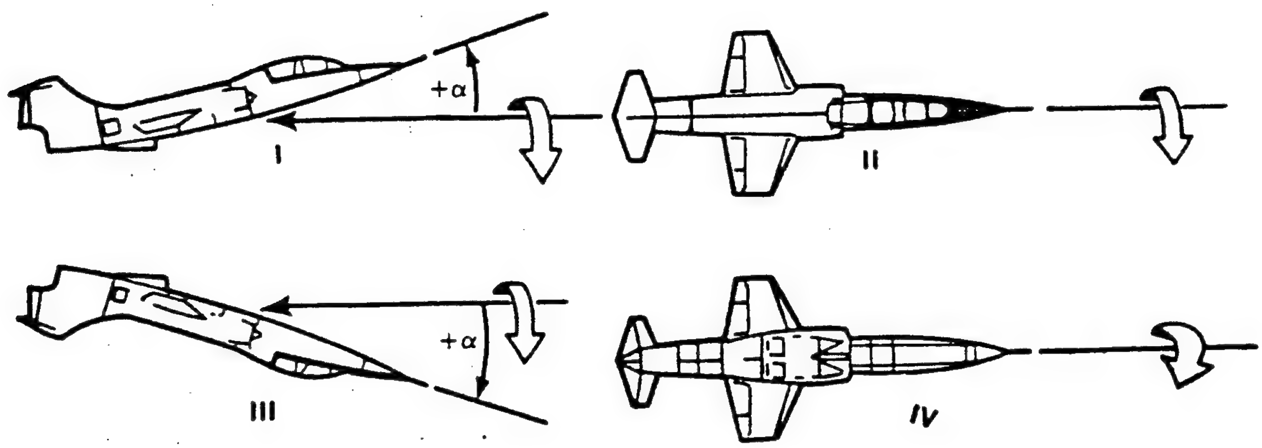


FIGURE 9.8. NO KINEMATIC COUPLING. ROLLING OF AN AIRCRAFT WITH INFINITELY LARGE STABILITY OR NEGLIGIBLE INERTIA IN PITCH AND YAW

In this case, the aircraft will roll about its aerodynamic axis, and no interchange of α or β will occur.

Since aircraft do not have infinitely large inertia or stability, neithe of these extremes can occur. Some combination of these effects will always result during a roll. The amount of kinematic coupling will depend upon the relative values of C_{n_β} and C_{m_α} and roll rates. This can be shown with two empirical relationships:

$$\dot{\alpha} = -Kp\beta \tag{9.16}$$

$$\dot{\beta} = Kp\alpha \tag{9.17}$$

These relationships show that any roll rate will cause an interchange of α and β. The exact amount depends on the magnitude of K which is determined by the relative values of the moments of inertia and C_{m_α} and C_{n_β}. It can also be seen that for a given aircraft, the rate of interchange of α and β depends on the roll rate. The higher the roll rate, the greater the kinematic coupling. As roll rate increases, a point is reached where the stability of

the aircraft is insufficient to counter the α and β build up. This divergence could ultimately result in departure from controlled flight. This point is of special interest to designers and is often the subject of an in-depth mathematical analysis. Although little can be determined from Equations 9.16 and 9.17, they provide a basis for showing how an aircraft's dynamic response can be used to make some rough predictions about the kinematic coupling characteristics. It has been shown in dynamics that the natural frequency of the short period mode is a function of C_{m_α}. Likewise the natural frequency of the Dutch Roll mode is a function of C_{n_β}.

Assume that an aircraft is rolled at a rate that creates a disturbance in β at a rate equal to the maximum rate that the natural aircraft stability can damp out the disturbance. Thus,

$$\dot{\beta} = Kp\alpha = f\,(\omega_n)_{\text{Dutch Roll}} \tag{9.18}$$

In this case, there would be no buildup of β, and a condition of neutral stability in yaw would result. However, if the roll rate were increased slightly above this value, then successively larger increases in β would occur and divergence would result. This analysis can also be followed through for an initial disturbance in α. It is not important which diverges first, α or β, since any divergence about one axis will quickly drive the other divergent. As a matter of interest however, supersonically C_{n_β} decreases more rapidly than C_{m_α} and therefore, most modern aircraft will diverge in yaw first, supersonically.

It can be shown on an analog computer that when $C_{m_\alpha} = C_{n_\beta}$ a stable condition will exist at all roll rates. This is often referred to as a "tuned condition", and is a possible dodge for an aircraft designer to use in a critical flight area. However, it is difficult to capitalize on this occurrence because of the wide variation of the stability derivatives with Mach.

It may be that an aircraft will possess stability parameters such that roll coupling problem exists at a given roll rate. However, if a relative long time is required before large values of α and β are generated, then t aircraft may be rolled at the maximum value by restricting the aircraft to c 360° roll. In this situation, the aircraft is diverging during the roll, t at such a slow rate that by the time the aircraft has rolled 360°, the maxim allowable α or β of the aircraft has not been exceeded.

9.5 AUTOROTATIONAL ROLLING

It has been shown that during rolling maneuvers, large angles of atta and sideslip may occur as a result of inertial and kinematic coupling. F some aircraft, certain conditions of α and β will produce a rolling mome that is in the same direction as the roll. If this moment is equal or great than the moment created by roll damping, the airplane will continue uncommanded roll. In some cases, it may not be possible to stop the aircra from rolling, although full lateral control is held against the ro direction. This is known as autorotational rolling or "auto roll". There a various conditions that can cause auto roll. It can occur at a positive negative angle of attack with any combination of sideslip angle. It is high dependent on aerodynamic design. However, flight control and stabili augmentation systems can also have a large effect. Auto roll is normal caused by the development of sideslip due to kinematic or inertial coupli and the effect of C_{1_β} once this sideslip has developed. On some aircraft wi highly augmented flight control systems, an auto roll may result from contr inputs commanded by the system itself.

A good example of auto roll occurs in the F-104 at negative angles attack. For analysis sake, let us assume the aircraft is rolled to the righ In this case the negative α is converted into negative β (refer to Figure 9. III and IV). The vertical stabilizer for the F-104 is highly effectiv therefore the $-\beta$ develops a significant rolling moment to the right whi reinforces the rolling motion. Since the F-104 is a fuselage loaded aircraf the rolling motion causes the airplane to pitch down. This increases the and further complicates the problem. If allowed to continue, this moti

could diverge until the aircraft departs from controlled flight. If an auto roll of this type were to begin, the pilot should pull back on the stick to make α positive. With $+\alpha$, kinematic coupling will tend to decrease the roll rate.

Although no analysis of the effects of augmented flight control systems (SAS, CAS, etc.) will be presented here, note that these types of systems are prone to cause auto roll tendencies. Rate feedbacks are hard to tailor to improve handling qualities throughout the flight regime without adversely affecting roll coupling tendencies somewhere in that regime. It is up to the flight test pilot and engineer to accurately predict where problems may exist and thoroughly investigate these areas.

9.6 CONCLUSIONS

As an aircraft's inertias are disproportionately increased in relation to its aerodynamic stabilities in pitch and yaw, the aircraft will be liable to pitching and yawing motions during rolling maneuvers. The more typical case is a divergence in yaw by virtue of an inadequate value of C_{n_β}.

The peak loads resulting from roll coupling generally increase in proportion to the initial incidence of the inertial axis and progressively with the duration of the roll and the rapidity of aileron application at the beginning and the end of the maneuver. The most severe cases naturally should be expected in a flight regime of low C_{n_β} and high dynamic pressures.

The rolling pull-out maneuver in a high performance aircraft is especially dangerous. It combines many unfavorable features: high speed, hence high roll rate capability; high acceleration which favors poor coordination and inadvertent excitation of transients by the pilot; and high dynamic pressures which at large values of α and β may break the aircraft.

Most high performance aircraft incorporate roll rate limiters in addition to angular damping augmentors. In these aircraft, a lateral control with enough power for low speed is almost certain to be too powerful for high speeds. Fortunately, limiters of various kinds are not too difficult to incorporate in a fully powered control system.

It is obvious that flight testing in suspected regions of roll coupli warrants a cautious methodical approach and must be accompanied by thorou computer studies that stay current with the flight test data. The only w that the pilot can discover the exact critical roll limit in flight is when exceeds it, which is obviously not the approach to take. Because of thi flight tests are generally discontinued when computer studies indicate th the next data point may be "over the line".

The following example is cited. The Bell X-2 rocket ship was launch from its mother ship at Edwards in 1956. The pilot flew a perfect profil but the rocket engine burned a few critical seconds longer than the enginee predicted, resulting in a greater speed (Mach 3.2) and greater altitu (119,800 feet) than planned. Unknown to the pilot, he was progressive running out of directional stability. When he was over the point at which had preplanned to start his turn toward Roger's Dry Lake he actuated h controls. The X-2 went divergent with a resultant loss of control. T accident investigation revealed the cause to be a greater loss in direction stability than planned, resulting in divergent roll coupling.

A combination of reasonable piloting restrictions coupled with increas directional stability has provided the solution to roll coupling problems the present generation of aircraft. The problem is one of understanding sin a thinking pilot would no more exceed the roll limitations imposed on aircraft than he would the structural "G" limitations.

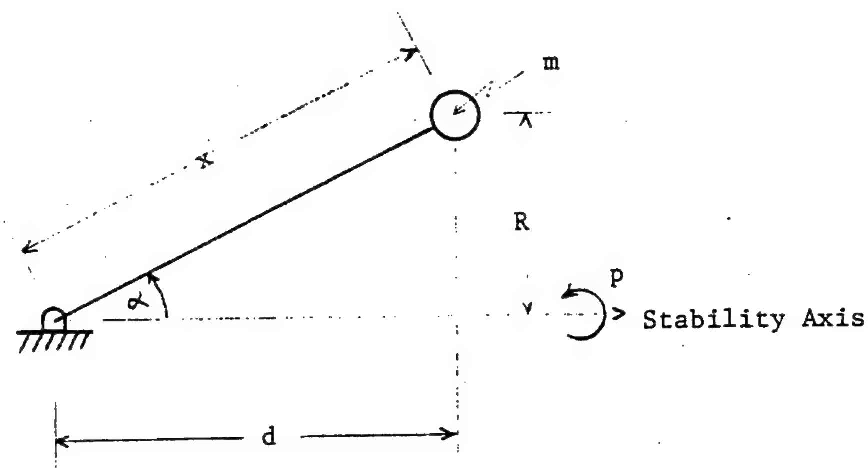
Besides pilot education, some other methods to eliminate roll coupli divergence are:

1. Roll Rate Limiters
2. Angular Damping Augmentors
3. Placarded Roll Limits, such as:
    a. "G" limits
    b. Total allowable roll at maximum rate
    c. Altitude limits
    d. Mach limits
    e. Flap position limits

PROBLEMS

9.1 Define "Divergence in roll coupling".

9.2 Assuming the aircraft rolls about the stability axis, derive an expression for pitching moment as a function of mass (m), roll rate (p), distance (x) and angle of attack (α).



9.3 Differentiate between "inertial coupling" and "the I_{xz} effect".

9.4 What two characteristics of the century series aircraft make them m
susceptible to roll coupling than World War II fighters?

9.5 Define aerodynamic (kinematic) coupling.

9.6 Give two causes of autorotational rolling.

Advanced Problems

9.7 Given the following expressions taken from AFFTC-TR-79-18 (F-16 High Angle-of-Attack Report).

$$\dot{\alpha} = q - \tan\beta(p\cos\alpha + r\sin\alpha) + \frac{Z}{m\,V\cos\alpha\cos\beta} + \frac{r\tan\alpha\tan\beta}{m\,V\cos\beta}$$

$$\dot{\beta} = p\sin\alpha - r\cos\alpha + \frac{Y}{m\,V\cos\beta}$$

assume small angles (α, β) and negligible forces (Y, Z); show that

$$\dot{\alpha} = q - p\beta$$
$$\dot{\beta} = p\alpha - r$$

Use the expressions for α and β from 9.7 and the expressions below:
(Assume a "principal axes" system)

$$G_y = \dot{q} I_y - rp (I_z - I_x) + (p^2 - r^2) I_{xz}$$

$$G_z = \dot{r} I_z - pq (I_x - I_y) + (qr - \dot{p}) I_{xz}$$

$$M = C_{m_\alpha} \alpha q S\bar{c} \qquad\qquad N = C_{n_\beta} \beta q S\bar{c}$$

Consider that for neutral divergence stability $\dot{p} = \dot{r} = \dot{\alpha} = \dot{\beta} = 0$. Show that the critical roll rate for pitch and yaw divergence is

for pitch divergence: $$p^2 = \frac{C_{m_\alpha} q S\bar{c}}{I_x - I_z}$$

for yaw divergence: $$p^2 = \frac{C_{n_\beta} q S\bar{c}}{I_y - I_x}$$

ANSWERS

9.1. Divergence in roll coupling is a departure from the intended flight path that will result in either loss of control or structural failure.

9.2 Centrifugal force (C.F.) = $\frac{mV_{tan}^2}{R}$

$V_{tan} = \omega R$

ω is about the x axis, therefore $\omega = p$

$V_{tan} = p\ R$

$C.F. = \frac{mp^2 R^2}{R}$

$C.F. = mp^2 R$

moment (M) = force x distance

M = C.F. x d

$M = mp^2\ Rd$

using geometry: $\sin \alpha = \frac{R}{x}$

$\cos \alpha = \frac{d}{x}$

$M = mp^2\ (x \sin \alpha)\ (x \cos \alpha)$

$M = mp^2\ x^2 \sin \alpha \cos \alpha$

but $\sin \alpha \cos \alpha = \frac{1}{2} \sin 2\alpha$

$M = \frac{1}{2}\ mp^2\ x^2 \sin 2\alpha$

9.3 Both inertial coupling and the I_{xz} effect occur because the roll axis the inertial axis are not coincident. Inertial coupling assumes the airc rolls about the aerodynamic axis. The I_{xz} effect assumes the aircraft r about the x axis (body axis). Depending on the relative orientation of inertial, x and aerodynamic axes, the I_{xz} effect may increase or decrease effect of inertial coupling.

9.4 Equation 9.2: $\dot{q} = \frac{\Sigma M}{I_y} - pr \left(\frac{I_x - I_z}{I_y} \right) - (p^2 - r^2) \frac{I_{xz}}{I_y}$

Fuselage loading (decreasing I_x) and high roll rates (increasing p) cause an uncontrollable pitch acceleration.

9.5 Aerodynamic or kinematic coupling is the actual interchange of α an during a rolling maneuver.

9.6 1) Large angles of attack and sideslip building up as a result inertial and kinematic coupling (F-104 at negative angles of attack).

2) Augmented flight controls, especially rate feedbacks (F-15 CAS).

9.7 $\dot{\alpha} = q - \tan\beta\,(p\cos\alpha + r\sin\alpha) + \dfrac{Z}{mV\cos\alpha\cos\beta} + \dfrac{r\tan\alpha\tan\beta}{mV\cos\beta}$

$\dot{\beta} = p\sin\alpha - r\cos\alpha + \dfrac{Y}{mV\cos\beta}$

assume small angles and negligible forces:

$\therefore$ $\sin\alpha \approx \alpha$, $\cos\alpha \approx 1$, $\tan\alpha \approx \alpha$

$\sin\beta \approx \beta$, $\cos\beta \approx 1$, $\tan\beta \approx \beta$

$\alpha \times \beta \approx 0$ (small x small = very small)

$Y = Z = 0$

substituting assumptions:

$$\dot{\alpha} = q - \beta\,(p + r\,\alpha) + \cancel{\frac{0}{mV}}^{\,0} + \cancel{\frac{r\,\alpha\beta}{mV}}^{\,0}$$

$$\dot{\alpha} = q - p\beta - r\,\cancel{\alpha\beta}^{\,0}$$

$$\dot{\alpha} = q - p\,\beta$$

$$\dot{\beta} = p\,\alpha - r + \cancel{\frac{0}{mV}}^{\,0}$$

$$\dot{\beta} = p\,\alpha - r$$

9.8 $G_y = \dot{q}\, I_y - rp\,(I_z - I_x) + (p^2 - r^2)\, I_{xz}$

$G_z = \dot{r}\, I_z - pq\,(I_x - I_y) + (qr - \dot{p})\, I_{xz}$

Assume principal axes system $\rightarrow I_{xz} = 0$

Given : $M = C_{m_\alpha}\, \alpha\, q\, S\bar{c}$

$N = C_{n_\beta}\, \beta q\, S\bar{c}$

substituting:

$$C_{m_\alpha}\, \alpha q S\bar{c} = \dot{q} I_y - rp\,(I_z - I_x)$$

$$C_{n_\beta}\, \beta q S\bar{c} = \dot{r}\, I_z - pq\,(I_x - I_y)$$

rearranging to solve for p:

$$\frac{C_{m_\alpha}\, \alpha q S\bar{c} - \dot{q}\, I_y}{r(I_z - I_x)} = -p \quad , \quad \frac{C_{n_\beta}\, \beta q\, S\bar{c} - \dot{r}\, I_z}{q(I_x - I_y)} = -p$$

$$\frac{C_{m_\alpha}\, \alpha q S\bar{c} - \dot{q}\, I_y}{r(I_x - I_z)} = p \quad , \quad \frac{C_{n_\beta}\, \beta q\, S\bar{c} - \dot{r}\, I_z}{q(I_y - I_x)} = p$$

Given: $\dot{\alpha} = q - p\beta$

$\dot{\beta} = p\alpha - r$

solving for q, r:

$$q = \dot{\alpha} + p\beta$$

$$r = p\alpha - \dot{\beta}$$

assuming $\dot{\alpha} = \dot{\beta} = \dot{q} = \dot{r} \approx 0$:

$$q = p\beta, \; r = p\alpha$$

$$\frac{C_{m_\alpha} \alpha q S\bar{c}}{r(I_x - I_z)} = p \, , \quad \frac{C_{n_\beta} \beta q S\bar{c}}{q(I_y - I_x)} = p$$

substituting values for r,q:

$$\frac{C_{m_\alpha} \alpha q S\bar{c}}{p\alpha(I_x - I_z)} = p \, , \quad \frac{C_{n_\beta} \beta q S\bar{c}}{p\beta(I_y - I_x)} = p$$

$$\frac{C_{m_\alpha} q S\bar{c}}{I_x - I_z} = p^2 \, , \quad \frac{C_{n_\beta} q S\bar{c}}{I_y - I_x} = p^2$$